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Sharp forms of Nevanlinna error terms in differential equations

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Abstract

Sharp versions of some classical results in differential equations are given. Main results consists of a Clunie and a Mohon'ko type theorems, both with sharp forms of error terms. The sharpness of these results is discussed and some applications to nonlinear differential equations are given in the conluding remarks.

Moreover, a short introduction on the connection between Nevanlinna theory and number theory, as well as on their relation to differential equations, is given. In addition, a brief review on the recent developments in the field of sharp error term analysis is presented.

1 Introduction

Rolf Nevanlinna's theory of value distribution is undoubtedly one of the great mathematical discoveries of the twentieth century. Nevanlinna laid the foundations of the theory in a 100 pages article appeared in 1925 [15]. This remarkable contribution was later on described by Hermann Weyl as

> *...one of the few great mathematical events in our century* [24].

The only remaining big open question, proposed and partially solved by Nevanlinna himself, was for a long time the *Nevanlinna inverse problem*, which is essentially a problem of finding a meromorphic function with prescribed deficient values. This problem was solved in 1976 by David Drasin, who used quasiconformal mappings to construct the desired function [5]. Undoubtedly this result was a substantial contribution to the theory, and some would even go so far as to say that Drasin's work finally completed Nevanlinna theory. But, no doubt, the richness of value distribution theory goes much further than that. Not only it has numerous applications in the fields of differential and functional equations, but there is a profound relation between Nevanlinna theory and number theory, which also extends to the theory of differential equations. In this paper we will concentrate on this deep connection between these three theories.

The present paper is organized as follows. We start by recalling the necessary notation in Section 2. We then continue by giving a short introduction to the deep connection between Nevanlinna theory and number theory in Section 3. This is by no means complete review of the topic, and, therefore we refer to [1], [3], [18] and [23] for a more comprehensive survey on the connection between there two theories. Then, in Section 4, we give a brief review on the recent developments in the sharp error term analysis and recall how this research

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is connected to the relation between Nevanlinna theory and number theory. In Section 5 we discuss how the theory of differential equations is related to the above topics. We also recall an important auxiliary result which we will use to prove our main results in Section 7. Sections 6–8 are the main sections of the present paper. They include the statement of our results in Section 6, the proofs of theorems in Section 7 and concluding remarks in Section 8.

2 Prerequisites

We use the standard notation of Nevanlinna theory, in other words $m(r,f)$, $N(r,f)$ and $T(r,f)$ denote the *proximity function*, the *counting function* and the *characteristic function* of f, respectively. See for example [7] and [11] for explicit definitions and basic results of Nevanlinna theory.

Consider an algebraic differential equation

$$P\left(z, w, w', \ldots, w^{(n)}\right) = 0 \tag{2.1}$$

with meromorphic coefficients. We may write (2.1) in the form

$$\sum_{\lambda=(j_0,\ldots,j_n)\in I} a_\lambda(z) w^{j_0}(w')^{j_1}\cdots(w^{(n)})^{j_n} = 0, \tag{2.2}$$

where I is a finite set of *multi-indices* λ, and a_λ are meromorphic functions for all $\lambda \in I$. While considering equation (2.2), coefficients a_λ are often chosen to satisfy certain growth conditions, for instance $T(r, a_\lambda) = S(r,f)$ for $\lambda \in I$. This will not, however, be necessary for the main results of the present paper. The *degree* of a single term in (2.2) is defined by

$$d(\lambda) := j_0 + j_1 + \cdots + j_n,$$

and its *weight* by

$$w(\lambda) := j_1 + 2j_2 + \cdots + nj_n.$$

Furthermore, we define the degree, $d(P)$, and the weight, $w(P)$, of the differential polynomial P as the maximal degree and the maximal weight of all terms of P, respectively. Moreover, given a multi-index $\lambda = (j_0, \ldots, j_n)$, we denote

$$|\lambda| := j_0 + \ldots + j_n.$$

Finally, if a set A satisfies $A \subset [0, 2\pi]$, then $|A| := \int_A \frac{dt}{2\pi}$.

The following, slightly modified version of the classical Jensen inequality is needed to prove an important auxiliary result below.

Theorem A (Jensen) *Let (X,μ) be a measure space such that $\mu(X) = \delta$, where $0 < \delta \leq 1$. Let χ be a concave function on the interval $(a, +\infty)$, where $a \geq -\infty$. Then, for any integrable real valued function f with $a < f(x) < +\infty$ for almost all x in X, we have*

$$\int_X \chi(f(x))d\mu(x) \leq \delta \cdot \chi\left(\frac{1}{\delta}\int_X f(x)d\mu(x)\right).$$

For a detailed proof of this refinement, see [2, Theorem 4.1].

Theorem A has the following useful corollary, which we name here as lemma. The proof is almost identical to that of [2, Corollary 4.2], but since the assertion is somewhat more general, we include the details of the proof here for convenience.

Lemma B *Let $f(z)$ be a meromorphic function in $\{z : |z| < R\}$, where $0 < R \leq \infty$, let $H \subset [0, 2\pi]$ and let $0 < \alpha < 1$. Then for $0 < r < R$,*

$$\int_H \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \frac{1}{\alpha} \left(\log^+ \int_H \left| f(re^{i\theta}) \right|^\alpha \frac{d\theta}{2\pi} + e^{-1} \right).$$

Proof. First define

$$E := \left\{ \theta \in H : \left| f(re^{i\theta}) \right| > 1 \right\},$$

and $F := H \backslash E$. Then, by Theorem A,

$$\begin{aligned}
\int_H \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} &= \frac{1}{\alpha} \int_H \log^+ \left| f(re^{i\theta}) \right|^\alpha \frac{d\theta}{2\pi} \\
&= \frac{1}{\alpha} \int_E \log \left| f(re^{i\theta}) \right|^\alpha \frac{d\theta}{2\pi} + \frac{1}{\alpha} \int_F \log^+ \left| f(re^{i\theta}) \right|^\alpha \frac{d\theta}{2\pi} \\
&\leq \frac{|E|}{2\pi\alpha} \log \left(\frac{2\pi}{|E|} \int_E \left| f(re^{i\theta}) \right|^\alpha \frac{d\theta}{2\pi} \right) \\
&\leq \frac{1}{\alpha} \left(\log^+ \left(\int_H \left| f(re^{i\theta}) \right|^\alpha \frac{d\theta}{2\pi} \right) + \frac{|E|}{2\pi} \log \frac{2\pi}{|E|} \right).
\end{aligned}$$

Since $x \log(x^{-1}) \leq e^{-1}$ for all $0 < x < 1$, the assertion follows. □

3 A connection between Nevanlinna theory and number theory

What could be described as the core of Nevanlinna theory is the Second main theorem, which is a deep result concerning the value distribution of meromorphic functions. It contains Picard's theorem,

> *a transcendental meromorphic function assumes infinitely often all values in the plane except at most two* [19],

as a special case, and, in its general form, is stated as follows [15]:

Theorem C (Nevanlinna) *Let f be a non-constant meromorphic function, let $q \geq 2$ and let $a_1, \ldots, a_q \in \mathbb{C}$ be distinct points. Then*

$$m(r, f) + \sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) \leq 2T(r, f) - N_1(r, f) + S(r, f),$$

where $N_1(r, f)$ is positive and given by

$$N_1(r, f) = N\left(r, \frac{1}{f'}\right) + 2N(r, f) - N(r, f').$$

We now recall another classical and deep result, this time from number theory. The proof of the following Thue-Siegel-Roth theorem [17] yielded Klaus Roth a Fields medal.

Table 1: Vojta's Dictionary

Value Distribution	**Diophantine Approximation**
A non-constant meromorphic function	An infinite $\{x\}$ in a number field F
A radius r	An element x of F
A finite measure set E of radii	A finite subset of $\{x\}$
An angle θ	An embedding $\sigma : F \hookrightarrow \mathbb{C}$
The modulus $\lvert f(re^{i\theta})\rvert$	The modulus $\lvert x\rvert_\sigma = \lvert\sigma(x)\rvert$
The order of a pole: $\mathrm{ord}_z f$	The order of x in the prime ideal: $\mathrm{ord}_\wp x$
$\log \frac{r}{\lvert z\rvert}$	$[F_\wp : \mathbb{Q}_p] \log p$
Characteristic function: $T(r,f) = m(r,f) + N(r,f)$	Logarithmic height: $h(x) = \frac{1}{[F:\mathbb{Q}]} \sum_{\sigma:F\hookrightarrow\mathbb{C}} \log^+ \lvert x\rvert_\sigma + N(x,F)$
Mean proximity function: $m\left(r, \frac{1}{f-a}\right) = \int_0^{2\pi} \log^+ \left\lvert \frac{1}{f(re^{i\theta})-a} \right\rvert \frac{d\theta}{2\pi}$	Mean proximity function: $m\left(x, \frac{1}{F-a}\right) = \sum_{\sigma:F\hookrightarrow\mathbb{C}} \log^+ \left\lvert \frac{1}{x-a} \right\rvert_\sigma$
Counting function: $N\left(r, \frac{1}{f-a}\right) = \sum_{0<\lvert z\rvert<r} \mathrm{ord}_z^+(f-a) \log \frac{r}{\lvert z\rvert} + \mathrm{ord}_0^+ \log r$	Counting function: $N\left(x, \frac{1}{F-a}\right) = \frac{1}{[F:\mathbb{Q}]} \sum_{\wp\in F} \mathrm{ord}_\wp^+(x-a)[F_\wp : \mathbb{Q}_p] \log p$
First main theorem: $T(r,f) = T\left(r, \frac{1}{f-a}\right) + O(1)$	Height property: $N\left(x, \frac{1}{F-a}\right) + m\left(x, \frac{1}{F-a}\right) = h(x) + O(1)$
Weaker Second main theorem: $(q-2)T(r,f) - \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) \leq \varepsilon T(r,f)$	Roth's theorem: $(q-2)h(x) - \sum_{j=1}^{q} N\left(x, \frac{1}{F-a_j}\right) \leq \varepsilon h(x)$

Theorem D (Roth) *Let α be an algebraic irrational number, and let $\varepsilon > 0$. Then there are only finitely many solutions p/q to*

$$\left\lvert \alpha - \frac{p}{q} \right\rvert \leq \frac{1}{q^{2+\varepsilon}},$$

where p and q are relatively prime integers.

At the first glance Second main theorem and Roth's theorem, like Nevanlinna theory and number theory in general, seem to have very little, or nothing, to do with each other. However, Charles Osgood observed that this is quite not the case. He realized that the occurrence of the number two in both of these classical results is by no means an accident. Osgood was the first to draw attention to the formal connections between Nevanlinna theory and number theory, Roth's theorem and Second main theorem being just one example [16]. Paul Vojta made this connection deeper and more extensive by introducing so called *Vojta's dictionary*, where he associated the terminology in Nevanlinna theory and Diophantine approximation theory in an explicit manner [23]. This approach is applied also in Ru's monograph [18], which offers extensive correspondence tables, in addition to Vojta's dictionary, between results and conjectures in Nevanlinna theory and Diophantine approximation theory. We are

settled with giving a few passages from Vojta's dictionary in Table 1, just to briefly illustrate the basic idea. For the complete representation, including a description of the relevant terminology, we refer to [23] and [18].

What still remains as an 'ultimate challenge', which would really complete the analogue between these two fields of mathematics, is to prove results in number theory, for instance Roth's theorem, by using methods from Nevanlinna theory. The aim is roughly the following: First take a result in Nevanlinna theory, then switch the corresponding terms of Nevanlinna theory with the concepts in number theory by using Vojta's dictionary, and finally follow the reasoning in the proof of the original result. The result of this procedure would be a theorem in number theory. This description is, of course, a crude simplification of the real situation. However, since Nevanlinna theory seems to be somewhat ahead of number theory in the sense of the correspondence described above, this approach would yield strong results in number theory, which so far remain unproved, including the famous *abc*-conjecture due to Masser and Oesterlé. Unfortunately, the dictionary does not extend to all concepts needed to follow the above algorithm. In particular, no counterpart for the derivative of a meromorphic function has yet been discovered. Therefore it is, so far, impossible to translate any proofs from Nevanlinna theory to number theory.

4 Sharp error term analysis

Serge Lang observed in the beginning of 1990's that the analogy between Nevanlinna theory and Diophantine approximation theory motivates the search for a sharp form of the error term in Nevanlinna's main theorems [12]. Lang's perception led to extensive research activity. Joseph Miles obtained a sharp form of the lemma on the logarithmic derivative both in the complex plane and in the unit disc [13]. He also showed that his results were essentially the best possible. His results were studied further by Marcus Jankowski [10], who showed that a slightly smaller error term may be obtained, but with a cost of a larger exceptional set. Moreover, Aimo Hinkkanen proved [9] a version of the Second main theorem where the error term was sharp in the sense of [13].

One of the fundamental results used in the sharp error term analysis is the following estimate by Gol'dberg and Grinshtein [6]:

$$m\left(r, \frac{f'}{f}\right) \leq \log^+\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right) + 5.8501, \tag{4.1}$$

where $0 < r < \rho < R$ and f is a meromorphic function in the disc $|z| < R$ such that $f(0) = 1$. Inequality (4.1) has been used, in one way or another, in most papers involving sharp error term analysis. The constant term 5.8501 was later on improved to 5.3078 by Djamel Benbourenane and the author [2]. Moreover, inequality (4.1) was generalized to higher order derivatives in a paper by Janne Heittokangas et al. [8], where one of the main results was the following:

Theorem E *Let f be a meromorphic function in $\mathbb{C}$ such that $f^{(j)}$ does not vanish identically, and let k and j be integers such that $k > j \geq 0$. Then there exists an $r_0 > 1$ such that*

$$m\left(r, \frac{f^{(k)}}{f^{(j)}}\right) \leq (k-j)\log^+\left(\frac{T(\rho, f)}{r}\frac{\rho}{\rho - r}\right) + \log\frac{k!}{j!} + (k-j)5.3078 \tag{4.2}$$

for all $r_0 < r < \rho < \infty$.

Analogous result was shown to be true also in the unit disc. It was also shown in [8] that the best possible constant in inequality (4.2) is greater or equal to $(k-j)\log\frac{\pi}{e}$. This obviously gives also a limit to how much the constant in (4.1) can be further improved. These sharpness considerations were accomplished by means of a surprisingly simple auxiliary function

$$f(z) = e^{z^n},$$

which satisfies

$$m\left(r, \frac{f^{(k)}}{f^{(j)}}\right) \geq (k-j)\log^+\left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho-r}\right) + (k-j)\log\frac{\pi}{e} - \varepsilon(r,\rho)$$

for all sufficiently large r, where ρ is set to be $\rho = \frac{n}{n-1}r$, $n \in \mathbb{N}\setminus\{1\}$. The expression $\varepsilon(r,\rho)$ tends to zero as r and n both approach to infinity. We summarize the above observations as follows:

Proposition 4.1 *Let $0 < r < \rho < R$. The smallest possible constant κ, such that*

$$m\left(r, \frac{f'}{f}\right) \leq \log^+\left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho-r}\right) + \kappa \tag{4.3}$$

for all meromorphic functions f in the disc $|z| < R$, satisfies

$$0.1447 < \log\frac{\pi}{e} \leq \kappa < 5.3078.$$

It is somewhat surprising that the small constant κ cannot be absorbed into the other, much larger term on the right hand side of inequality (4.3). It would be interesting to see what the reason for this is. It is also possible that this fact has a counterpart in number theory as well. But first, one would need to find what the best possible constant is. This, however, may prove to be a difficult task, and, since the (partially numerical) methods used in [6] and [2] are quite technical and complicated, it is likely that completely new ideas are needed to find the desired constant.

5 Differential equations

Another connection between Nevanlinna theory and number theory can be found through the theory of differential equations. On one hand, Nevanlinna theory offers an efficient way to study meromorphic solutions of differential equations, since it allows the study of the properties of a solution without knowing its explicit form. A good example of this efficacy is Kosaku Yosida's simple alternative proof for classical Malmquist's theorem [25]. On the other hand, differential equations may be used as a tool to study transcendental numbers. This approach has been used, for instance, by Andrei Shidlovskii in his monograph [20]. Therefore differential equations can be understood as a 'bridge' between Nevanlinna theory and transcendental number theory.

We now state two further examples from differential equations, where Nevanlinna theory has been successfully utilized to obtain accurate information about the meromorphic solutions. We start with the Clunie lemma [4], which is an efficient tool to analyze the density of poles of solutions of certain kind of algebraic differential equations.

Lemma F (Clunie) *Let f be a transcendental meromorphic solution of the differential equation*
$$f^n P(z,f) = Q(z,f),$$
where $P(z,f)$ and $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda : \lambda \in I\}$, such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is $\leq n$, then

$$m\left(r, P(z,f)\right) = S(r,f).$$

The second result is the following Mohon'ko's theorem [14, Theorem 6]. It can be used, similarly as Clunie's lemma may be used to study pole distribution, to study value distribution of meromorphic solutions of differential equations.

Theorem G (Mohon'ko) *Let $P(z,u)$ be a polynomial in u and its derivatives with meromorphic coefficients, say $\{a_\lambda : \lambda \in I\}$, such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$, and let $u = f$ be a transcendental meromorphic solution of the differential equation*

$$P(z,u) = 0.$$

If $P(z,0) \not\equiv 0$, then
$$m\left(r, \frac{1}{f}\right) = S(r,f).$$

The fact that connects these two classical results is that both of their proofs are based on a natural generalization
$$m\left(r, \frac{f^{(n)}}{f}\right) = S(r,f)$$
of the lemma on the logarithmic derivative. Therefore, one might expect that inequality (4.2) would yield even stronger results. Indeed this is true, but to obtain more natural results somewhat different auxiliary lemma is needed, although it is in the same spirit as inequality (4.2). The following lemma was proved in [8].

Lemma H *Let f be a non-polynomial meromorphic function in $\mathbb{C}$. Let k and j be integers satisfying $k > j \geq 0$, α and β be constants satisfying $0 < \alpha(k-j) < 1$ and $0 < \beta < 1$, and let $\varepsilon > 0$. Then there exists an $r_0(\varepsilon) > 1$ such that, for all $r_0 < r < \rho < \infty$,*

$$\int_0^{2\pi} \left| \frac{f^{(k)}(re^{i\theta})}{f^{(j)}(re^{i\theta})} \right|^\alpha \frac{d\theta}{2\pi} \leq \left(C\big(\alpha(k-j),\beta\big) + \varepsilon\right) \left(\frac{k!}{j!}\right)^\alpha \left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho - r}\right)^{\alpha(k-j)},$$

where

$$C(a,b) = \left(\frac{2}{1-b}\right)^a + \frac{\sec\left(\frac{a\pi}{2}\right)}{b^a}\left(4 + \left(2^{\frac{1+a}{1-a}} + 2^{\frac{2+a}{1-a}}\right)^{1-a}\right). \tag{5.1}$$

Remark. It was also calculated in [2] that

$$\frac{\log(C(\alpha,\beta)+\varepsilon) + e^{-1}}{\alpha} + \varepsilon < 5.3078,$$

when $\alpha = 0.815508$, $\beta = 0.845890$, and $\varepsilon > 0$ is chosen to be sufficiently small.

Lemma H and the remark above will be repeatedly applied in the proofs of our main results.

6 Statement of results

Understanding the whole structure behind the connections between these three seemingly disconnected fields of mathematics, differential equations, Nevanlinna theory and number theory, could be described as assembling a complicated puzzle without the help of an exemplar. Some of the pieces are already in place, and so we are beginning to see how the big picture looks like. However, large part of the work is still undone, especially in understanding how differential equations fit into this picture.

We will make here, best to our knowledge, the first attempt to gain more information about these connections by close examination of error terms in differential equations. We give versions of the two classical results stated in Section 5 with sharp forms of error terms. We start with a Clunie type result. Although Clunie modestly called his original result lemma, we label ours as a theorem since it is one of the main results of the present paper.

Theorem 6.1 *Let f be a transcendental meromorphic solution of the differential equation*

$$f^n P(z,f) = Q(z,f),$$

where

$$P(z,f) = \sum_{\lambda \in I} a_\lambda f^{i_0} (f')^{i_1} \cdots (f^{(m)})^{i_m}$$

and

$$Q(z,f) = \sum_{\mu \in J} b_\mu f^{j_0} (f')^{j_1} \cdots (f^{(k)})^{j_k}.$$

If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is $\leq n$, then there exists r_0 such that

$$\begin{aligned} m\left(r, P(z,f)\right) \leq & \big(w(P)+w(Q)\big) \log^+ \left(\frac{T(\rho,f)}{r} \frac{\rho}{\rho - r} \right) + \max_{\lambda \in I} m(r, a_\lambda) \\ & + \max_{\mu \in J} m(r, b_\mu) + d(P)\log m! + d(Q) \log k! + \log^+ \mathrm{card}(I) \\ & + \log^+ \mathrm{card}(J) + \big(w(P)+w(Q)\big) 5.3078 \end{aligned}$$

for all $r_0 < r < \rho < \infty$.

The next result is a sharp form of Mohon'ko's theorem.

Theorem 6.2 *Let f be a transcendental meromorphic solution of the differential equation*

$$P(z,f) = 0, \tag{6.1}$$

where

$$P(z,f) = \sum_{\lambda \in I} a_\lambda f^{i_0} (f')^{i_1} \cdots (f^{(m)})^{i_m}.$$

If $a_0 := P(z,0) \not\equiv 0$, then there exists r_0 such that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) \leq & \, w(P) \log^+ \left(\frac{T(\rho,f)}{r} \frac{\rho}{\rho - r} \right) + \max_{\substack{\lambda \in I \\ |\lambda| \geq 1}} m(r, a_\lambda) + m\left(r, \frac{1}{a_0}\right) \\ & + d(P) \log m! + \log^+(\mathrm{card}(I) - 1) + w(P) 5.3078 \end{aligned}$$

for all $r_0 < r < \rho < \infty$.

7 Proofs of theorems

The basic ideas behind the proofs of our results are adopted from the original works of Clunie and Mohon'ko. These ideas are complemented by more recent results taking the form of the error term into account.

The proof of Theorem 6.1

Defining

$$\begin{aligned} E_1 &= \big\{\varphi \in [0,2\pi) : |f(re^{i\varphi})| < 1\big\} \\ E_2 &= [0,2\pi) \setminus E_1 \end{aligned} \tag{7.1}$$

and denoting

$$P(z) := P(z,f) =: \sum_{\lambda\in I} a_\lambda P_\lambda(z), \tag{7.2}$$

we consider the proximity function $m(r,P)$ in two parts:

$$m(r,P) = \int_{E_1} \log^+ |P(re^{i\varphi})| \frac{d\varphi}{2\pi} + \int_{E_2} \log^+ |P(re^{i\varphi})| \frac{d\varphi}{2\pi}.$$

We start by looking at $P_\lambda(z)$ in the set E_1. Let $\alpha \in (0, \frac{1}{w(\lambda)})$ and $\lambda = (i_0, i_1, \ldots, i_m)$. Then, by the Hölder inequality and Lemma H, there exists r_1 such that

$$\begin{aligned} \int_{E_1} |P_\lambda(re^{i\varphi})|^\alpha \frac{d\varphi}{2\pi} &\le \int_0^{2\pi} \left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right|^{\alpha i_1} \cdots \left|\frac{f^{(m)}(re^{i\varphi})}{f(re^{i\varphi})}\right|^{\alpha i_m} \frac{d\varphi}{2\pi} \\ &\le \left(\int_0^{2\pi} \left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right|^{\alpha w(\lambda)} \frac{d\varphi}{2\pi}\right)^{\frac{i_1}{w(\lambda)}} \cdots \left(\int_0^{2\pi} \left|\frac{f^{(m)}(re^{i\varphi})}{f(re^{i\varphi})}\right|^{\frac{\alpha w(\lambda)}{m}} \frac{d\varphi}{2\pi}\right)^{\frac{m i_m}{w(\lambda)}} \\ &\le \left(\big(C\big(\alpha w(\lambda), \beta\big) + \varepsilon\big) \left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho - r}\right)^{\alpha w(\lambda)}\right)^{\frac{i_1}{w(\lambda)}} \\ &\quad \cdots \left(\big(C\big(\alpha w(\lambda), \beta\big) + \varepsilon\big) (m!)^{\frac{\alpha w(\lambda)}{m}} \left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho - r}\right)^{\alpha w(\lambda)}\right)^{\frac{m i_m}{w(\lambda)}} \\ &= \big(C\big(\alpha w(\lambda), \beta\big) + \varepsilon\big)\, 1 \cdot (2!)^{\alpha i_2} \cdots (m!)^{\alpha i_m} \left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho - r}\right)^{\alpha w(\lambda)} \end{aligned} \tag{7.3}$$

for all $r_1 < r < \rho < \infty$. Now, remembering notation (7.2), we get

$$\begin{aligned} \int_{E_1} \log^+ |P(re^{i\varphi})| \frac{d\varphi}{2\pi} &= \int_{E_1} \log^+ \left|\sum_{\lambda\in I} a_\lambda(re^{i\varphi}) P_\lambda(re^{i\varphi})\right| \frac{d\varphi}{2\pi} \\ &\le \int_{E_1} \log^+ \left(\operatorname{card}(I) \max_{\lambda\in I} \big|a_\lambda(re^{i\varphi}) P_\lambda(re^{i\varphi})\big|\right) \frac{d\varphi}{2\pi} \\ &\le \max_{\lambda\in I}\left(\int_{E_1} \log^+ \big|P_\lambda(re^{i\varphi})\big| \frac{d\varphi}{2\pi} + m(r,a_\lambda)\right) + \log^+ \operatorname{card}(I). \end{aligned} \tag{7.4}$$

Hence, by Lemma B, inequality (7.3) and the remark below Lemma H, we obtain

$$
\begin{aligned}
\int_{E_1} \log^+ |P(re^{i\varphi})| \frac{d\varphi}{2\pi} &\leq \max_{\lambda \in I} \left(\frac{1}{\alpha} \left(\log^+ \int_{E_1} \left|P_\lambda(re^{i\varphi})\right|^\alpha \frac{d\varphi}{2\pi} + e^{-1} \right)\right) \\
&\quad + \max_{\lambda\in I} m(r, a_\lambda) + \log^+ \operatorname{card}(I) \\
&\leq \max_{\lambda \in I} \frac{1}{\alpha} \log^+ \left(\big(C\big(\alpha w(\lambda), \beta\big) + \varepsilon\big) \, 1 \cdot (2!)^{\alpha i_2} \cdots (m!)^{\alpha i_m} \left(\frac{T(\rho, f)}{r} \frac{\rho}{\rho - r} \right)^{\alpha w(\lambda)} \right) \\
&\quad + \frac{e^{-1}}{\alpha} + \max_{\lambda\in I} m(r, a_\lambda) + \log^+ \operatorname{card}(I) \\
&\leq w(P) \log^+ \left(\frac{T(\rho, f)}{r} \frac{\rho}{\rho - r} \right) + \max_{\lambda\in I} m(r, a_\lambda) + d(P) \log^+ m! + \log^+ \operatorname{card}(I) \\
&\quad + w(P) \max_{\lambda\in I} \frac{\log^+ \big(C\big(\alpha w(\lambda), \beta\big) + \varepsilon\big) + e^{-1}}{\alpha w(\lambda)} \\
&\leq w(P) \log^+ \left(\frac{T(\rho, f)}{r} \frac{\rho}{\rho - r} \right) + \max_{\lambda\in I} m(r, a_\lambda) + d(P) \log^+ m! + \log^+ \operatorname{card}(I) \\
&\quad + w(P) 5.3078
\end{aligned} \tag{7.5}
$$

for all $r_1 < r < \rho < \infty$.

To consider the set E_2, recall that we assumed $j_0 + \cdots + j_k \leq n$ for all $\mu = (j_0, \cdots, j_k) \in J$. Therefore,

$$
\begin{aligned}
\int_{E_2} \log^+ |P(re^{i\varphi})| \frac{d\varphi}{2\pi} &= \int_{E_2} \log^+ \left| \frac{1}{f(re^{i\varphi})^n} \sum_{\mu\in J} b_\mu(re^{i\varphi}) f'(re^{i\varphi})^{j_1} \cdots f^{(m)}(re^{i\varphi})^{j_k} \right| \frac{d\varphi}{2\pi} \\
&\leq \int_{E_2} \log^+ \left| \sum_{\mu\in J} b_\mu(re^{i\varphi}) \left(\frac{f'(re^{i\varphi})}{f(re^{i\varphi})} \right)^{j_1} \cdots \left(\frac{f^{(k)}(re^{i\varphi})}{f(re^{i\varphi})} \right)^{j_k} \right| \frac{d\varphi}{2\pi} \\
&\leq \max_{\mu\in J} \left(\int_{E_2} \log^+ \left| \left(\frac{f'(re^{i\varphi})}{f(re^{i\varphi})} \right)^{j_1} \cdots \left(\frac{f^{(k)}(re^{i\varphi})}{f(re^{i\varphi})} \right)^{j_k} \right| \frac{d\varphi}{2\pi} + m(r, b_\mu) \right) \\
&\quad + \log^+ \operatorname{card}(J).
\end{aligned} \tag{7.6}
$$

By continuing with essentially identical reasoning as in inequalities (7.3), (7.4) and (7.5), we finally conclude that there is a constant r_2 such that

$$
\begin{aligned}
\int_{E_2} \log^+ |P(re^{i\varphi})| \frac{d\varphi}{2\pi} \leq w(Q) \log^+ \left(\frac{T(\rho, f)}{r} \frac{\rho}{\rho - r} \right) + \max_{\mu\in J} m(r, b_\mu) \\
+ d(Q) \log^+ k! + \log^+ \operatorname{card}(J) + w(Q) 5.3078
\end{aligned} \tag{7.7}
$$

for all $r_2 < r < \rho < \infty$.

The assertion follows by setting $r_0 = \max\{r_1, r_2\}$, and by combining estimates (7.5) and (7.7). □

Proof of Theorem 6.2

We may write equation (6.1) in the form

$$P(z, f) = a_0(z) + Q(z) = 0,$$

where $a_0(z) = P(z,0) \not\equiv 0$ by assumption, and where

$$Q(z) = \sum_{\substack{\lambda \in I \\ |\lambda| \geq 1}} a_\lambda f^{i_0} (f')^{i_1} \cdots (f^{(m)})^{i_m}.$$

Recalling (7.1), we consider the situation again in the two sets E_1 and E_2 separately. Clearly the integral $m(r, 1/f)$ vanishes on E_2. On the other hand, in E_1,

$$\frac{1}{|f|} \left| f^{i_0} (f')^{i_1} \cdots (f^{(m)})^{i_m} \right| \leq \left| \frac{f'}{f} \right|^{i_1} \cdots \left| \frac{f^{(m)}}{f} \right|^{i_m}$$

for each term of $Q(z)$. Therefore,

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &= \int_{E_1} \log^+ \left| \frac{1}{f(re^{i\varphi})} \right| \frac{d\varphi}{2\pi} \\ &= \int_{E_1} \log^+ \left| \frac{a_0(re^{i\varphi})}{f(re^{i\varphi})} \frac{1}{a_0(re^{i\varphi})} \right| \frac{d\varphi}{2\pi} \\ &= \int_{E_1} \log^+ \left| \frac{Q(re^{i\varphi})}{f(re^{i\varphi})} \frac{1}{a_0(re^{i\varphi})} \right| \frac{d\varphi}{2\pi} \\ &\leq \int_{E_1} \log^+ \left| \frac{Q(re^{i\varphi})}{f(re^{i\varphi})} \right| \frac{d\varphi}{2\pi} + m\left(r, \frac{1}{a_0}\right) \\ &\leq \int_{E_1} \log^+ \left((\mathrm{card}(I) - 1) \max_{\substack{\lambda \in I \\ |\lambda| \geq 1}} \left| \frac{a_\lambda(re^{i\varphi}) f(re^{i\varphi})^{i_0} f'(re^{i\varphi})^{i_1} \cdots f^{(m)}(re^{i\varphi})^{i_m}}{f(re^{i\varphi})} \right| \right) \frac{d\varphi}{2\pi} \\ &\quad + m\left(r, \frac{1}{a_0}\right) \\ &\leq \max_{\substack{\lambda \in I \\ |\lambda| \geq 1}} \int_{E_1} \log^+ \left| \frac{f'(re^{i\varphi})}{f(re^{i\varphi})} \right|^{i_1} \cdots \left| \frac{f^{(m)}(re^{i\varphi})}{f(re^{i\varphi})} \right|^{i_m} \frac{d\varphi}{2\pi} + \max_{\substack{\lambda \in I \\ |\lambda| \geq 1}} m(r, a_\lambda) \\ &\quad + m\left(r, \frac{1}{a_0}\right) + \log^+(\mathrm{card}(I) - 1). \end{aligned}$$

Again, by applying the reasoning used in inequalities (7.3), (7.4) and (7.5), we conclude that there is a constant r_0 such that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) \leq w(P) \log^+ \left(\frac{T(\rho, f)}{r} \frac{\rho}{\rho - r} \right) + \max_{\substack{\lambda \in I \\ |\lambda| \geq 1}} m(r, a_\lambda) + m\left(r, \frac{1}{a_0}\right) \\ + d(P) \log^+ m! + \log^+(\mathrm{card}(I) - 1) + w(P) 5.3078 \end{aligned}$$

for all $r_0 < r < \rho < \infty$. □

8 Concluding remarks

The obvious question which we have not addressed so far, is whether or not our main estimates are, in one sense or another, the best possible. We conclude the present paper by discussing this question, and looking at some applications of our theorems. Since Theorem 6.1 can only be applied to nonlinear differential equations, we start with the simplest nonlinear equation, which is the Riccati equation.

Example 8.1 All meromorphic solutions of the Riccati differential equation

$$w' = a(z)w^2 + b(z)w + c(z), \tag{R}$$

with rational coefficients $a(z)$, $b(z)$ and $c(z)$, are of finite order of growth, see [11] for example. Assume that f is such a solution. Then, by Theorem 6.1,

$$\limsup_{r\longrightarrow+\infty} \frac{m(r,f)}{\log r} \leq \max\{0, \rho(f)-1\} + \mathrm{di}^+(a) + \max\{\mathrm{di}^+(b), \mathrm{di}^+(c)\},$$

where, given a rational function $r = p/q$, where p and q are polynomials,

$$\mathrm{di}(r) := \deg(p) - \deg(q)$$

denotes the *degree* of a rational function r *at infinity*, and

$$\mathrm{di}^+(r) := \max\{0, \mathrm{di}(r)\}.$$

Similarly, by Theorem 6.2,

$$\limsup_{r\longrightarrow+\infty} \frac{m\left(r, \frac{1}{f-q}\right)}{\log r} \leq \max\{0, \rho(f)-1\} + \max\left\{\mathrm{di}^+(a), \mathrm{di}^+(b)\right\} + \mathrm{di}^+\left(\frac{1}{c}\right),$$

where $q \in \mathbb{C}$. In particular, since $\tan z$ is a solution of

$$w' = w^2 + 1,$$

we have

$$\limsup_{r\longrightarrow+\infty} \frac{m(r, \tan z)}{\log r} = \limsup_{r\longrightarrow+\infty} \frac{m\left(r, \frac{1}{\tan z}\right)}{\log r} = 0,$$

which is, of course, a well known fact.

In fact, Theorem 6.1 correctly yields

$$m(r, \tan z) = O(1),$$

and similarly, by Theorem 6.2, we obtain

$$m\left(r, \frac{1}{\tan z}\right) = O(1).$$

Unfortunately, this is not enough to demonstrate the sharpness of the main estimates. Indeed, since the weight of the Riccati equation is one, it is not likely to yield any further information. So, in order to get better examples, we must turn our attention to higher order nonlinear differential equations, the next obvious choice being the Painlevé differential equations.

Example 8.2 Consider the first, second and the fourth Painlevé differential equations

$$w'' = 6w^2 + z, \tag{P_I}$$

$$w'' = 2w^3 + zw + \alpha, \tag{P_{II}}$$

$$ww'' = \frac{1}{2}(w')^2 + \frac{3}{2}w^4 + 4zw^3 + 2(z^2 - \beta)w^2 + \gamma, \qquad (P_{IV})$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. It is well known that all solutions of these equations are meromorphic, and of finite order of growth. It is also known that

$$m(r, f) = O(\log r),$$

if f is a solution of any of the equations (P_I), (P_{II}) or (P_{IV}). Now, by Theorem 6.1, we have in fact that

$$m(r, f_I) \leq 3 \log r + O(1), \qquad (8.1)$$

where f_I is a solution of (P_I). Similarly

$$m(r, f_{II}) \leq 4 \log r + O(1) \qquad (8.2)$$

and

$$m(r, f_{IV}) \leq 6 \log r + O(1), \qquad (8.3)$$

where f_{II} and f_{IV} are solutions of (P_{II}) and (P_{IV}), respectively. While deriving inequalities (8.2), (8.2) and (8.3), we have used, in addition to Theorem 6.1, the fact that $\rho(f_I) = 5/2$, $\rho(f_{II}) \leq 3$ and $\rho(f_{IV}) \leq 4$, see [21] or [22].

Finding out whether or not the estimates (8.1), (8.2) and (8.3) are sharp, would possibly give an answer to the question about sharpness of our main estimates. However, since dealing with the Painlevé transcendents is highly complicated issue in general, this is not an easy task, and, indeed is beyond the scope of this paper. We are settled with leaving this matter as an open question.

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Erratum:

In the above article there is an error in the proofs of Theorem 6.1 and 6.2. The expression "$\max_{\lambda\in I}$" in inequalities (7.4) and (7.5), and in the inequality on page 11, should be replaced by "$\sum_{\lambda\in I}$", and similarly "$\max_{\mu\in J}$" in inequalities (7.6) and (7.7) should be "$\sum_{\mu\in J}$". This modification has a slight effect on the final form of Theorems 6.1 and 6.2. The correct statements of these theorems are as follows.

Theorem 6.1. Let f be a transcendental meromorphic solution of the differential equation

$$f^n P(z,f) = Q(z,f),$$

where

$$P(z,f) = \sum_{\lambda\in I} P_\lambda(z,f) = \sum_{\lambda\in I} a_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(m)})^{i_m}$$

and

$$Q(z,f) = \sum_{\mu\in J} Q_\lambda(z,f) = \sum_{\mu\in J} b_\mu f^{j_0}(f')^{j_1}\cdots(f^{(k)})^{j_k}.$$

If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is $\leq n$, then there exists r_0 such that

$$\begin{aligned} m\left(r, P(z,f)\right) \leq{}& \left(\sum_{\lambda\in I} w(P_\lambda) + \sum_{\mu\in J} w(Q_\mu)\right)\log^+\left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho-r}\right) + \sum_{\lambda\in I} m(r,a_\lambda) \\ &+ \sum_{\mu\in J} m(r,b_\mu) + \sum_{\lambda\in I} d(P_\lambda)\log m! + \sum_{\mu\in J} d(Q_\mu)\log k! + \log^+\operatorname{card}(I) \\ &+ \log^+\operatorname{card}(J) + \left(\sum_{\lambda\in I} w(P_\lambda) + \sum_{\mu\in J} w(Q_\mu)\right)5.3078 \end{aligned}$$

for all $r_0 < r < \rho < \infty$.

Here $d(P_\lambda) = i_0+\cdots+i_m$ is the *degree* of $P_\lambda(z,f) = a_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(m)})^{i_m}$, and $w(P_\lambda) = i_1 + 2i_2 + \cdots + mi_m$ denotes the *weight* of $P_\lambda(z,f)$.

Theorem 6.2. Let f be a transcendental meromorphic solution of the differential equation

$$P(z,f) = 0,$$

where

$$P(z,f) = \sum_{\lambda\in I} P_\lambda(z,f) = \sum_{\lambda\in I} a_\lambda f^{i_0}(f')^{i_1}\cdots(f^{(m)})^{i_m}.$$

If $a_0 := P(z,0) \not\equiv 0$, then there exists r_0 such that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) \leq{}& \sum_{\lambda\in I} w(P_\lambda)\log^+\left(\frac{T(\rho,f)}{r}\frac{\rho}{\rho-r}\right) + \sum_{\substack{\lambda\in I\\ |\lambda|\geq 1}} m(r,a_\lambda) + m\left(r,\frac{1}{a_0}\right) \\ &+ \sum_{\lambda\in I} d(P_\lambda)\log m! + \log^+(\operatorname{card}(I)-1) + \sum_{\lambda\in I} w(P_\lambda)5.3078 \end{aligned}$$

for all $r_0 < r < \rho < \infty$.

Theorems 6.1 and 6.2 were applied to find upper bounds for the proximity functions of the first, second and fourth Painlevé transcendents. Due to the above mentioned error, these estimates (8.1) – (8.3) are also incorrect. The correct estimates are

$$m(r, f_I) \leq 4 \log r + O(1), \tag{8.1}$$

$$m(r, f_{II}) \leq 5 \log r + O(1), \tag{8.2}$$

$$m(r, f_{IV}) \leq 15 \log r + O(1). \tag{8.3}$$

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